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SIGNED WORDS AND PERMUTATIONS, IV; FIXED AND PIXED POINTS

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Von Jacobs hat er die Statur,
Des Rechnens ernstes Führen,
Von Lottärchen die Frohnatur
und Lust zu diskretieren.

To Volker Strehl, a dedication à la Goethe,
on the occasion of his sixtieth birthday.

Abstract

The flag-major index "fmaj" and the classical length function "ℓ" are used to construct two q-analogs of the generating polynomial for the hyperoctahedral group B_n by number of positive and negative fixed points (resp. pixed points). Specializations of those q-analogs are also derived dealing with signed derangements and desarrangements, as well as several classical results that were previously proved for the symmetric group.

1. Introduction

The statistical study of the hyperoctahedral group B_n, initiated by Reiner ([Re93a], [Re93b], [Re93c], [Re95a], [Re95b]), has been rejuvenated by Adin and Roichman [AR01] with their introduction of the *flag-major index*, which was shown [ABR01] to be equidistributed with the *length function*. See also their recent papers on the subject [ABR05], [ReRo05]. It then appeared natural to extend the numerous results obtained for the symmetric group $\mathfrak{S}_n$ to the groug B_n. Furthermore, flag-major index and length function become the true q-analog makers needed for calculating various multivariable distributions on B_n.

In the present paper we start with a generating polynomial for B_n by a three-variable statistic involving the number of fixed points (see formula (1.3)) and show that there are two ways of q-analogizing it, by using the flag-major index on the one hand, and the length function, on the other hand. As will be indicated, the introduction of an extra variable Z makes it possible to specialize all our results to the symmetric group. Let us first give the necessary notations.

Let B_n be the hyperoctahedral group of all *signed permutations* of order n. The elements of B_n may be viewed as words $w = x_1x_2\cdots x_n$,

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where each x_i belongs to $\{-n,\ldots,-1,1,\ldots,n\}$ and $|x_1||x_2|\cdots|x_n|$ is a permutation of $12\ldots n$. The *set* (resp. the *number*) of *negative* letters among the x_i's is denoted by $\operatorname{Neg} w$ (resp. $\operatorname{neg} w$). A *positive fixed point* of the signed permutation $w = x_1x_2\cdots x_n$ is a (positive) integer i such that $x_i = i$. It is convenient to write $\bar{i} := -i$ for each integer i. Also, when A is a set of integers, let $\overline{A} := \{\bar{i} : i \in A\}$. If $x_i = \bar{i}$ with i positive, we say that $\bar{i}$ is a *negative fixed point* of w. The set of all positive (resp. negative) fixed points of w is denoted by $\operatorname{Fix}^+ w$ (resp. $\operatorname{Fix}^- w$). Notice that $\operatorname{Fix}^- w \subset \operatorname{Neg} w$. Also let

$$\operatorname{fix}^+ w := \#\operatorname{Fix}^+ w;\quad \operatorname{fix}^- w := \#\operatorname{Fix}^- w. \tag{1.1}$$

There are $2^n n!$ signed permutations of order n. The symmetric group $\mathfrak{S}_n$ may be considered as the subset of all w from B_n such that $\operatorname{Neg} w = \emptyset$.

The purpose of this paper is to provide *two q-analogs* for the polynomials $B_n(Y_0,Y_1,Z)$ defined by the identity

$$\sum_{n\ge 0}\frac{u^n}{n!}B_n(Y_0,Y_1,Z) = \bigl(1-u(1+Z)\bigr)^{-1}\times\frac{\exp(u(Y_0+Y_1Z))}{\exp(u(1+Z))}. \tag{1.2}$$

When $Z=0$, the right-hand side becomes $(1-u)^{-1}\exp(uY_0)/\exp(u)$, which is the exponential generating function for the generating polynomials for the groups $\mathfrak{S}_n$ by number of fixed points (see [Ri58], chap. 4). Also, by identification, $B_n(1,1,1) = 2^n n!$ and it is easy to show (see Theorem 1.1) that $B_n(Y_0,Y_1,Z)$ is in fact the generating polynomial for the group B_n by the three-variable statistic $(\operatorname{fix}^+, \operatorname{fix}^-, \operatorname{neg})$, that is,

$$B_n(Y_0,Y_1,Z) = \sum_{w\in B_n} Y_0^{\operatorname{fix}^+ w}\, Y_1^{\operatorname{fix}^- w}\, Z^{\operatorname{neg} w}. \tag{1.3}$$

Recall the traditional notations for the q-ascending factorials

$$(a;q)_n := \begin{cases} 1, & \text{if } n=0;\\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n\ge 1;\end{cases} \tag{1.4}$$
$$(a;q)_\infty := \prod_{n\ge 1}(1-aq^{n-1});$$

for the q-multinomial coefficients

$$\begin{bmatrix} n \\ m_1,\ldots,m_k\end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_{m_1}\cdots(q;q)_{m_k}}\quad (m_1+\cdots+m_k=n); \tag{1.5}$$

and for the two q-exponentials (see [GaRa90, chap. 1])

$$e_q(u) = \sum_{n\ge 0}\frac{u^n}{(q;q)_n} = \frac{1}{(u;q)_\infty};\quad E_q(u) = \sum_{n\ge 0}\frac{q^{\binom{n}{2}}u^n}{(q;q)_n} = (-u;q)_\infty. \tag{1.6}$$

Our two q-analogs, denoted by ${}^{\ell}B_n(q,Y_0,Y_1,Z)$ and $B_n(q,Y_0,Y_1,Z)$, are respectively defined by the identities:

$$\sum_{n\ge 0}\frac{u^n}{(-Zq;q)_n\,(q;q)_n}\,{}^{\ell}B_n(q,Y_0,Y_1,Z) = \Bigl(1-\frac{u}{1-q}\Bigr)^{-1}\times(u;q)_\infty\Bigl(\sum_{n\ge 0}\frac{(-qY_0^{-1}Y_1Z;q)_n\,(uY_0)^n}{(-Zq;q)_n\,(q;q)_n}\Bigr); \tag{1.7}$$

$$\sum_{n\ge 0}\frac{u^n}{(q^2;q^2)_n}B_n(q,Y_0,Y_1,Z) = \Bigl(1-u\frac{1+qZ}{1-q^2}\Bigr)^{-1}\times\frac{(u;q^2)_\infty}{(uY_0;q^2)_\infty}\frac{(-uqY_1Z;q^2)_\infty}{(-uqZ;q^2)_\infty}. \tag{1.8}$$

Those two identities can be shown to yield (1.2) when $q=1$.

There is also a graded form of (1.8) in the sense that an extra variable t can be added to form a new polynomial $B_n(t,q,Y_0,Y_1,Z)$ with nonnegative integral coefficients that specializes into $B_n(q,Y_0,Y_1,Z)$ for $t=1$. Those polynomials are defined by the identity

$$\sum_{n\ge 0}(1+t)B_n(t,q,Y_0,Y_1,Z)\frac{u^n}{(t^2;q^2)_{n+1}} = \sum_{s\ge 0}t^s\Bigl(1-u\sum_{i=0}^{s}q^iZ^{\chi(i\ \mathrm{odd})}\Bigr)^{-1}\times\frac{(u;q^2)_{\lfloor s/2\rfloor+1}}{(uY_0;q^2)_{\lfloor s/2\rfloor+1}}\frac{(-uqY_1Z;q^2)_{\lfloor (s+1)/2\rfloor}}{(-uqZ;q^2)_{\lfloor (s+1)/2\rfloor}}, \tag{1.9}$$

where for each statement A we let $\chi(A)=1$ or 0 depending on whether A is true or not. The importance of identity (1.9) lies in its numerous specializations, as can be seen in Fig. 1.

The two q-extensions ${}^{\ell}B_n(q,Y_0,Y_1,Z)$ and $B_n(t,q,Y_0,Y_1,Z)$ being now defined, the program is to derive appropriate combinatorial interpretations for them. Before doing so we need have a second combinatorial interpretation for the polynomial $B_n(Y_0,Y_1,Z)$ besides the one mentioned in (1.3). Let $w=x_1x_2\cdots x_n$ be a word, all letters of which are integers without any repetitions. Say that w is a *desarrangement* if $x_1>x_2>\cdots>x_{2k}$ and $x_{2k}<x_{2k+1}$ for some $k\ge 1$. By convention, $x_{n+1}=\infty$. We could also say that the *leftmost trough* of w occurs at an *even* position. This notion was introduced by Désarménien [De84] and elegantly used in a subsequent paper [DeWa88]. Notice that there is no one-letter desarrangement. By convention, the empty word e is also a desarrangement.

Now let $w=x_1x_2\cdots x_n$ be a signed permutation. Unless w is increasing, there is always a nonempty right factor of w which is a desarrangement. It then makes sense to define w^d as the *longest* such a right factor. Hence, w admits a unique factorization $w=w^-w^+w^d$, called its *pixed*[(1)]

[(1)] "Pix," of course, must not be taken here for the abbreviated form of "pictures."

factorization, where w^- and w^+ are both *increasing*, the letters of w^- being *negative*, those of w^+ *positive* and where w^d is the longest right factor of w which is a desarrangement.

For example, the pixed factorizations of the following signed permutations are materialized by vertical bars: $w = \overline{5}\,\overline{2} \mid e \mid \overline{3}\,\overline{4}\,1$; $w = \overline{5} \mid e \mid \overline{2}\,\overline{3}\,1\,\overline{4}$; $w = \overline{5}\,\overline{3}\,\overline{2} \mid 1\,4 \mid e$; $\quad w = \overline{5}\,\overline{3} \mid 1 \mid 4\,2$; $\quad w = \overline{5}\,\overline{3} \mid e \mid 4\,1\,2$.

Let $w = w^- w^+ w^d$ be the pixed factorization of $w = x_1 x_2 \cdots x_n$. If $w^- = x_1 \cdots x_k$, $w^+ = x_{k+1} \cdots x_{k+l}$, define $\operatorname{Pix}^- w := \{x_1, \ldots, x_k\}$, $\operatorname{Pix}^+ w := \{x_{k+1}, \ldots, x_{k+l}\}$, $\operatorname{pix}^- w := \#\operatorname{Pix}^- w$, $\operatorname{pix}^+ w := \#\operatorname{Pix}^+ w$.

Theorem 1.1. *The polynomial* $B_n(Y_0, Y_1, Z)$ *defined by* (1.2) *admits the following two combinatorial interpretations:*

$$B_n(Y_0, Y_1, Z) = \sum_{w \in B_n} Y_1^{\operatorname{fix}^- w}\, Y_2^{\operatorname{fix}^+ w}\, Z^{\operatorname{neg} w} = \sum_{w \in B_n} Y_1^{\operatorname{pix}^- w}\, Y_2^{\operatorname{pix}^+ w}\, Z^{\operatorname{neg} w}.$$

Theorem 1.1 is proved in section 2. A bijection ϕ of B_n onto itself will be constructed that satisfies $(\operatorname{Fix}^-, \operatorname{Fix}^+, \operatorname{Neg})\, w = (\operatorname{Pix}^-, \operatorname{Pix}^+, \operatorname{Neg})\, \phi(w)$.

Let "ℓ" be the length function of B_n (see [Bo68, p. 7], [Hu90, p. 12] or the working definition given in (3.1)). As seen in Theorem 1.2, "ℓ" is to be added to the three-variable statistic $(\operatorname{pix}^+, \operatorname{pix}^-, \operatorname{neg})$ (and not to $(\operatorname{fix}^+, \operatorname{fix}^-, \operatorname{neg})$) for deriving the combinatorial interpretation of ${}^{\ell}B_n(q, Y_0, Y_1, Z)$. This theorem is proved in section 3.

Theorem 1.2. *For each* $n \geq 0$ *let* ${}^{\ell}B_n(q, Y_0, Y_1, Z)$ *be the polynomial defined in* (1.7)*. Then*

$$\text{(1.10)} \qquad {}^{\ell}B_n(q, Y_0, Y_1, Z) = \sum_{w \in B_n} q^{\ell(w)}\, Y_0^{\operatorname{pix}^+ w}\, Y_1^{\operatorname{pix}^- w}\, Z^{\operatorname{neg} w}.$$

The variables t and q which are added to interpret our second extension $B_n(t, q, Y_0, Y_1, Z)$ will carry the flag-descent number "fdes" and the flag-major index "fmaj." For each signed permutation $w = x_1 x_2 \cdots x_n$ the usual *number of descents* "des" is defined by $\operatorname{des} w := \sum_{i=1}^{n-1} \chi(x_i > x_{i+1})$, the *major index* "maj" by $\operatorname{maj} w := \sum_{i=1}^{n-1} i\, \chi(x_i > x_{i+1})$, the *flag descent number* "fdes" and the *flag-major index* "fmaj" by

$$\text{(1.11)} \qquad \operatorname{fdes} w := 2 \operatorname{des} w + \chi(x_1 < 0); \quad \operatorname{fmaj} w := 2 \operatorname{maj} w + \operatorname{neg} w.$$

Theorem 1.3. *For each* $n \geq 0$ *let* $B_n(t, q, Y_0, Y_1, Z)$ *be the polynomial defined in* (1.9)*. Then*

$$\text{(1.12)} \qquad B_n(t, q, Y_0, Y_1, Z) = \sum_{w \in B_n} t^{\operatorname{fdes} w}\, q^{\operatorname{fmaj} w}\, Y_0^{\operatorname{fix}^+ w}\, Y_1^{\operatorname{fix}^- w}\, Z^{\operatorname{neg} w}.$$

Theorem 1.3 is proved in Section 5 after discussing the combinatorics of the so-called *weighted signed permutations* in Section 4. Section 6 deals with numerous specializations of Theorem 1.2 and 1.3 obtained by taking numerical values, essentially 0 or 1, for certain variables. Those specializations are illustrated by the following diagram (Fig. 1). When $Z = 0$, the statistic "neg" plays no role and the signed permutations become plain permutations; the second column of the diagram is then mapped on the third one that only involves generating polynomials for $\mathfrak{S}_n$ or subsets of that group.

$$\begin{tikzcd}
D_n^B \arrow[r, "{(\mathrm{fmaj},\mathrm{fix}^-,\mathrm{neg})}"] & D_n^B(q,Y_1,Z) \arrow[d, "\mathrm{fdes}"] & D_n(q) \arrow[l, "\mathrm{neg}"'] \arrow[d, "\mathrm{des}"] & D_n \arrow[l, "\mathrm{maj}"'] \\
& D_n^B(t,q,Y_1,Z) \arrow[d, "\mathrm{fix}^+"] & D_n(t,q) \arrow[l, "\mathrm{neg}"'] \arrow[d, "\mathrm{fix}^+"] & \\
& \mathbf{B_n(t,q,Y_0,Y_1,Z)} & A_n(t,q,Y_0) \arrow[l, "\mathrm{neg}"'] & \\
& B_n(q,Y_0,Y_1,Z) \arrow[u, "\mathrm{fdes}"] & A_n(q,Y_0) \arrow[l, "\mathrm{neg}"'] \arrow[u, "\mathrm{des}"] & \\
B_n \arrow[r, "{(\mathrm{fix}^-,\mathrm{fix}^+,\mathrm{neg})}", "{(\mathrm{pix}^-,\mathrm{pix}^+,\mathrm{neg})}"'] & \mathbf{B_n(Y_0,Y_1,Z)} \arrow[u, "\mathrm{fmaj}"] \arrow[d, "\ell"] & A_n(Y_0) \arrow[l, "\mathrm{neg}"'] \arrow[u, "\mathrm{maj}"] \arrow[d, "\mathrm{inv}"] & \mathfrak{S}_n \arrow[l, "\mathrm{fix}^+"', "\mathrm{pix}^+"] \\
& {}^{\ell}\mathbf{B_n(q,Y_0,Y_1,Z)} & {}^{\mathrm{inv}}A_n(q,Y_0) \arrow[l, "\mathrm{neg}"'] & \\
K_n^B \arrow[r, "{(\ell,\mathrm{pix}^-,\mathrm{neg})}"'] & {}^{\ell}K_n^B(q,Y_1,Z) \arrow[u, "\mathrm{pix}^+"] & {}^{\mathrm{inv}}K_n(q) \arrow[l, "\mathrm{neg}"'] \arrow[u, "\mathrm{pix}^+"] & K_n \arrow[l, "\mathrm{inv}"', "\mathrm{imaj}"]
\end{tikzcd}$$

Fig. 1

The first (resp. fourth) column refers to specific subsets of B_n (resp. of $\mathfrak{S}_n$):

$$\begin{aligned}
D_n &:= \{w \in B_n : \mathrm{Fix}^+ w = \mathrm{Neg}\, w = \emptyset\};\\
K_n &:= \{w \in B_n : \mathrm{Pix}^+ w = \mathrm{Neg}\, w = \emptyset\};\\
D_n^B &:= \{w \in B_n : \mathrm{Fix}^+ w = \emptyset\};\\
K_n^B &:= \{w \in B_n : \mathrm{Pix}^+ w = \emptyset\}.
\end{aligned} \tag{1.13}$$

The elements of D_n are the classical *derangements* and provide the most natural combinatorial interpretations of the derangement numbers $d_n = \#D_n$ (see [Co70], p. 9–12). By analogy, the elements of D_n^B are called *signed derangements*. They have been studied by Chow [Ch06] in a recent note. The elements of K_n (resp. of K_n^B) are called *desarrangements* (resp.

signed desarrangements) of order n. When $Y_0=0$, the statistic fix^+ (resp. pix^+) plays no role. We can then calculate generating functions for signed (resp. plain) derangements or desarrangements, as shown in the first two and last rows. The initial polynomial, together with its two q-analogs are reproduced in boldface.

2. Proof of Theorem 1.1

As can be found in ([Co70], p. 9–12)), the generating function for the derangement numbers d_n $(n\ge 0)$ is given by

$$\sum_{n\ge 0} d_n \frac{u^n}{n!} = (1-u)^{-1}e^{-u}. \tag{2.1}$$

An easy calculation then shows that the polynomials $B_n(Y_0,Y_1,Z)$, introduced in (1.2), can also be defined by the identity

$$B_n(Y_0,Y_1,Z)=\sum_{i+j+k+l=n}\binom{n}{i,j,k,l} Y_0^i\,Y_1^j\,Z^{j+k}\,d_{k+l}\quad (n\ge 0). \tag{2.2}$$

For each signed permutation $w=x_1x_2\cdots x_n$ let $A:=\mathrm{Fix}^+ w$, $B:=\mathrm{Fix}^- w$, $C:=\mathrm{Neg}\, w\setminus \mathrm{Fix}^- w$, $D:=[n]\setminus(A\cup \overline{B}\cup\overline{C})$. Then $(A,\overline{B},\overline{C},D)$ is a sequence of disjoint subsets of integers, whose union is the interval $[n]:=\{1,2,\ldots,n\}$. Also the mapping τ defined by $\tau(\overline{j})=x_j$ if $\overline{j}\in C$ and $\tau(j)=x_j$ if $j\in D$ is a *derangement* of the set $C+D$. Hence, w is completely characterized by the sequence (A,B,C,D,τ). The generating polynomial for B_n by the statistic $(\mathrm{fix}^+,\mathrm{fix}^-,\mathrm{neg})$ is then equal to the right-hand side of (2.2). This proves the first identity of Theorem 1.1.

Each signed permutation $w=x_1x_2\cdots x_n$ can be characterized, either by the four-term sequence $(\mathrm{Fix}^+ w,\mathrm{Fix}^- w,\mathrm{Neg}\, w,\tau)$, as just described, or by $(\mathrm{Pix}^+ w,\mathrm{Pix}^- w,\mathrm{Neg}\, w, w^d)$, where w^d is the desarrangement occurring as the third factor in its pixed factorization. To construct a bijection ϕ of B_n onto B_n such that $(\mathrm{fix}^-,\mathrm{fix}^+,\mathrm{neg})\, w = (\mathrm{pix}^-,\mathrm{pix}^+,\mathrm{neg})\,\phi(w)$ and accordingly prove the second identity of Theorem 1.1, we only need a bijection $\tau\mapsto f(\tau)$, that maps each derangement τ onto a desarrangement $f(\tau)$ by rearranging the letters of τ. But such a bijection already exists. It is due to Désarménien (*op. cit.*). We describe it by means of an example.

Start with a derangement $\tau=\begin{pmatrix}1\,2\,3\,4\,5\,6\,7\,8\,9\\ 9\,7\,4\,3\,8\,2\,6\,5\,1\end{pmatrix}$ and express it as a product of its disjoint cycles: $\tau=(1\,9)(2\,7\,6)(3\,4)(5\,8)$. In each cycle, write the minimum in *second* position: $\tau=(9\,1)(6\,2\,7)(4\,3)(8\,5)$. Then, reorder the cycles in such a way that the sequence of those minima, when reading from left to right, is *decreasing*: $\tau=(8\,5)(4\,3)(6\,2\,7)(9\,1)$. The desarrangement $f(\tau)$ is derived from the latter expression by removing the parentheses: $f(\tau)=8\,5\,4\,3\,6\,2\,7\,9\,1$.

Let $(\mathrm{Fix}^+ w, \mathrm{Fix}^- w, \mathrm{Neg}\, w, \tau)$ be the sequence associated with the signed permutation w and let v^- (resp. v^-) be the *increasing* sequence of the elements of $\mathrm{Fix}^- w$ (resp. of $\mathrm{Fix}^+ w$). Then, $v^- \mid v^+ \mid f(\tau)$ is the pixed factorization of $v^- v^+ f(\tau)$ and we may define $\phi(w)$ by

$$\phi(w) := v^- v^+ f(\tau). \tag{2.3}$$

This defines a bijection of B_n onto itself, which has the further property:

$$(\mathrm{Fix}^-, \mathrm{Fix}^+, \mathrm{Neg})\, w = (\mathrm{Pix}^-, \mathrm{Pix}^+, \mathrm{Neg})\, \phi(w). \tag{2.4}$$

For instance, with $w = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 3\,\overline{2}\,8\,4\,5\,\overline{1}\,9\,\overline{6}\,7 \end{pmatrix}$ we have $v^+ = 4\,5$, $v^- = \overline{2}$, $\tau = \begin{pmatrix} \overline{1}\,3\,\overline{6}\,7\,8\,9 \\ 3\,8\,\overline{1}\,9\,\overline{6}\,7 \end{pmatrix} = (9\,7)(8\,\overline{6}\,\overline{1}\,3)$ and $f(\tau) = 9\,7\,8\,\overline{6}\,\overline{1}\,3$. Hence, the pixed factorization of $\phi(w)$ reads $\overline{2} \mid 4\,5 \mid 9\,7\,8\,\overline{6}\,\overline{1}\,3$ and $\phi(w) = \overline{2}\,4\,5\,9\,7\,8\,\overline{6}\,\overline{1}\,3$.

3. Proof of Theorem 1.2

The length function "ℓ" for B_n is expressed in many ways. We shall use the following expression derived by Brenti [Br94]. Let $w = x_1 x_2 \cdots x_n$ be a signed permutation; its *length* $\ell(w)$ is defined by

$$\ell(w) := \mathrm{inv}\, w + \sum_i |x_i|\, \chi(\, x_i < 0), \tag{3.1}$$

where "inv" designates the usual *number of inversions* for words:

$$\mathrm{inv}\, w := \sum_{1 \le i < j \le n} \chi(x_i > x_j).$$

The generating polynomial for K_n (as defined in (1.13)) by "inv" (resp. for D_n by "maj") is denoted by $K_n(q)$ (resp. $D_n(q)$). As was proved in [DeWa93] we have:

$$K_n(q) = D_n(q). \tag{3.2}$$

Also

$$\sum_{n \ge 0} \frac{u^n}{(q;q)_n} D_n(q) = \Bigl(1 - \frac{u}{1-q}\Bigr)^{-1} \times (u;q)_\infty, \tag{3.3}$$

as shown by Wachs [Wa90] in an equivalent form. Another expression for $D_n(q)$ will be derived in section 6, Proposition 6.2.

If A is a finite set of positive integers, let $\mathrm{tot}\, A$ denote the sum $\sum a$ ($a \in A$). For the proof of Theorem 1.2 we make use of the following classical result, namely that $q^{N(N+1)/2} {n \brack N}_q$ is equal to the sum $\sum q^{\mathrm{tot}\, A}$, where the sum is over all subsets A of cardinality N of the set $[n]$.

Remember that each signed permutation $w = x_1x_2\dots x_n$ is characterized by a sequence (A,B,C,D,τ), where $A=\mathrm{Pix}^+ w$, $B=\mathrm{Pix}^- w$, $C=\mathrm{Neg}\,w\setminus B$, $D=[n]\setminus(A\cup\overline{B}\cup\overline{C})$ and τ is a *desarrangement* of the set $C+D$. Let $\mathrm{inv}(B,C)$ be the number of pairs of integers (i,j) such that $i\in B$, $j\in C$ and $i>j$. As $\mathrm{inv}(B,C)=\mathrm{inv}(\overline{C},\overline{B})$, we have $\mathrm{inv}\,w=\mathrm{inv}(\overline{B},\overline{C})+\mathrm{inv}(A,D)+\#A\times\#C+\mathrm{inv}\,\tau$. From (3.1) it follows that

$$\begin{aligned}\ell(w)&=\mathrm{inv}\,w+\sum_{x_i<0}|x_i|=\mathrm{inv}\,w+\mathrm{tot}\,\overline{B}+\mathrm{tot}\,\overline{C}\\&=\mathrm{tot}\,\overline{B}+\mathrm{tot}\,\overline{C}+\mathrm{inv}(\overline{C},\overline{B})+\mathrm{inv}(A,D)+\#A\times\#C+\mathrm{inv}\,\tau.\end{aligned}$$

Denote the right-hand side of (1.10) by $G_n := G_n(q,Y_0,Y_1,Z)$. We will calculate $G_n(q,Y_0,Y_1,Z)$ by first summing over all sequences (A,B,C,D,τ) such that $\#A=i$, $\#B=j$, $\#C=k$, $\#D=l$. Accordingly, τ is a desarrangement of a set of cardinality $k+l$. We may write:

$$\begin{aligned}G_n&=\sum_{i+j+k+l=n}\ \sum_{(A,B,C,D)} q^{\mathrm{tot}\,\overline{B}+\mathrm{tot}\,\overline{C}+\mathrm{inv}(\overline{C},\overline{B})+\mathrm{inv}(A,D)+i\cdot k}\\&\qquad\qquad\times Y_0^iY_1^jZ^{j+k}\sum_{\tau\in K_{k+l}}q^{\mathrm{inv}\,\tau}\\&=\sum_{m+p=n}\ \sum_{\substack{j+k=m,\\ i+l=p}}\ \sum_{\substack{\#E=m,\\ F=[n]\setminus E}}\ \sum_{\substack{\overline{C}+\overline{B}=E,\\ A+D=F}} q^{\mathrm{tot}\,E+\mathrm{inv}(\overline{C},\overline{B})+\mathrm{inv}(A,D)+i\cdot k}\\&\qquad\qquad\times Y_0^i(Y_1Z)^jZ^kD_{k+l}(q)\\&=\sum_{m+p=n}\ \sum_{\substack{j+k=m,\\ i+l=p}} Y_0^i(Y_1Z)^j(Zq^i)^kD_{k+l}(q)\\&\qquad\qquad\times\sum_{\substack{\#E=m,\\ F=[n]\setminus E}}q^{\mathrm{tot}\,E}\sum_{\substack{\overline{C}+\overline{B}=E,\\ A+D=F}}q^{\mathrm{inv}(\overline{C},\overline{B})+\mathrm{inv}(A,D)}\\&=\sum_{m+p=n}\ \sum_{\substack{j+k=m,\\ i+l=p}} Y_0^i(Y_1Z)^j(Zq^i)^kD_{k+l}(q)\\&\qquad\qquad\times q^{m(m+1)/2}\begin{bmatrix}n\\ m\end{bmatrix}_q\begin{bmatrix}m\\ j,k\end{bmatrix}_q\begin{bmatrix}p\\ i,l\end{bmatrix}_q.\end{aligned}$$

Thus

$$G_n=\sum_{i+j+k+l=n}\begin{bmatrix}n\\ i,j,k,l\end{bmatrix}_q q^{\binom{j+k+1}{2}}Y_0^i(Y_1Z)^j(Zq^i)^kD_{k+l}(q). \tag{3.4}$$

Now form the factorial generating function

$$G(q,Y_0,Y_1,Z;u):=\sum_{n\ge0}\frac{u^n}{(-Zq;q)_n\,(q;q)_n}G_n(q,Y_0,Y_1,Z).$$

It follows from (3.4) that

$$G(q,Y_0,Y_1,Z;u)=\sum_{n\ge 0}\frac{1}{(-Zq;q)_n}\sum_{i+j+k+l=n} q^{\binom{j+k+1}{2}}\frac{(uY_0)^i}{(q;q)_i}\frac{(uY_1Z)^j}{(q;q)_j} \times u^{n-i-j}\frac{D_{k+l}(q)(Zq^i)^k}{(q;q)_k(q;q)_l}.$$

But $\binom{j+k+1}{2}=\binom{j+1}{2}+(j+1)k+\binom{k}{2}$. Hence

$$G(q,Y_0,Y_1,Z;u)=\sum_{n\ge 0}\frac{1}{(-Zq;q)_n}\sum_{m=0}^{n}\sum_{i+j=m} q^{\binom{j+1}{2}}\frac{(uY_0)^i}{(q;q)_i}\frac{(uY_1Z)^j}{(q;q)_j} \times \frac{u^{n-m}}{(q;q)_{n-m}}D_{n-m}(q)\sum_{k+l=n-m}\begin{bmatrix}n-m\\ k,l\end{bmatrix}_q (Zq^{m+1})^k q^{\binom{k}{2}}.$$

Now

$$(-Zq^{m+1};q)_{n-m}=\sum_{k+l=n-m}\begin{bmatrix}n-m\\ k,l\end{bmatrix}_q (Zq^{m+1})^k q^{\binom{k}{2}};$$

and

$$(-Zq;q)_n=(-Zq;q)_m\,(-Zq^{m+1};q)_{n-m}.$$

Hence

$$\begin{aligned}G(q,Y_0,Y_1,Z;u)&=\sum_{n\ge 0}\sum_{m=0}^{n}\frac{1}{(-Zq;q)_m}\sum_{i+j=m}\frac{(uY_0)^i}{(q;q)_i}q^{\binom{j+1}{2}}\frac{(uY_1Z)^j}{(q;q)_j} \times\frac{u^{n-m}}{(q;q)_{n-m}}D_{n-m}(q)\\ &=\Bigl(\sum_{n\ge 0}\frac{a_n\,u^n}{(-Zq;q)_n\,(q;q)_n}\Bigr)\Bigl(\sum_{n\ge 0}\frac{u^n}{(q;q)_n}D_n(q)\Bigr),\end{aligned}$$

with

$$\begin{aligned}a_n&=\sum_{i+j=n}\begin{bmatrix}n\\ i,j\end{bmatrix}_q Y_0^i q^{\binom{j}{2}}(qY_1Z)^j\\ &=Y_0^n\sum_{i+j=n}\begin{bmatrix}n\\ i,j\end{bmatrix}_q (qY_0^{-1}Y_1Z)^j q^{\binom{j}{2}}\\ &=Y_0^n(-qY_0^{-1}Y_1Z;q)_n.\end{aligned}$$

By taking (3.3) into account this shows that $G(q,Y_0,Y_1,Z;u)$ is equal to the right-hand side of (1.7) and then $G_n(q,Y_0,Y_1,Z)={}^{\ell}B_n(q,Y_0,Y_1,Z)$ holds for every $n\ge 0$. The proof of Theorem 1.2 is completed. By (3.4) we also conclude that the identity

$$(3.5)\quad {}^{\ell}B_n(q,Y_0,Y_1,Z)=\sum_{i+j+k+l=n}\begin{bmatrix}n\\ i,j,k,l\end{bmatrix}_q q^{\binom{j+k+1}{2}}Y_0^i(Y_1Z)^j(Zq^i)^kD_{k+l}(q)$$

is equivalent to (1.7). As its right-hand side tends to the right-hand side of (2.2) when $q \to 1$, we can then assert that (1.7) specializes into (1.2) for $q = 1$.

4. Weighted signed permutations

We use the following notations: if $c = c_1 c_2 \cdots c_n$ is a word, whose letters are nonnegative integers, let $\lambda(c) := n$ be the *length* of c, $\operatorname{tot} c := c_1 + c_2 + \cdots + c_n$ the *sum* of its letters and $\operatorname{odd} c$ the number of its *odd* letters. Furthermore, NIW_n (resp. $\mathrm{NIW}_n(s)$) designates the set of all *nonincreasing* words of length n, whose letters are nonnegative integers (resp. nonnegative integers at most equal to s). Also let $\mathrm{NIW}_n^e(s)$ (resp. $\mathrm{DW}_n^o(s)$) be the subset of $\mathrm{NIW}_n(s)$ of the nonincreasing (resp. strictly decreasing) words all letters of which are *even* (resp. *odd*).

Next, each pair $\binom{c}{w}$ is called a *weighted signed permutation* of order n if the four properties (*wsp1*)–(*wsp4*) hold:

(*wsp1*) c is a word $c_1 c_2 \cdots c_n$ from NIW_n;

(*wsp2*) w is a signed permutation $x_1 x_2 \cdots x_n$ from B_n;

(*wsp3*) $c_k = c_{k+1} \Rightarrow x_k < x_{k+1}$ for all $k = 1, 2, \ldots, n-1$;

(*wsp4*) x_k is positive (resp. negative) whenever c_k is even (resp. odd).

When w has no fixed points, either negative or positive, we say that $\binom{c}{w}$ is a *weighted signed derangement*. The set of weighted signed permutations (resp. derangements) $\binom{c}{w} = \binom{c_1 c_2 \cdots c_n}{x_1 x_2 \cdots x_n}$ of order n is denoted by WSP_n (resp. by WSD_n). The subset of all those weighted signed permutations (resp. derangements) such that $c_1 \le s$ is denoted by $\mathrm{WSP}_n(s)$ (resp. by $\mathrm{WSD}_n(s)$).

For example, the following pair

$$\binom{c}{w} = \left(\begin{array}{cc|c|ccc|ccc|c|cc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 10 & 10 & 9 & 7 & 7 & 7 & 4 & 4 & 4 & 3 & 2 & 2 & 1 \\ 1 & 2 & \overline{7} & \overline{6} & \overline{5} & \overline{4} & 3 & 8 & 9 & \overline{10} & 12 & 13 & \overline{11} \end{array}\right)$$

is a weighted signed permutation of order 13. It has four positive fixed points (1, 2, 8, 9) and two negative fixed points ($\overline{5}$, $\overline{10}$).

Proposition 4.1. *With each weighted signed permutation $\binom{c}{w}$ from the set $\mathrm{WSP}_n(s)$ can be associated a unique sequence $(i, j, k, \binom{c'}{w'}, v^e, v^o)$ such that*

(1) *i, j, k are nonnegative integers of sum n;*

(2) *$\binom{c'}{w'}$ is a weighted signed derangement from the set $\mathrm{WSD}_i(s)$;*

(3) *v^e is a nonincreasing word with even letters from the set $\mathrm{NIW}_j^e(s)$;*

(4) *v^o is a decreasing word with odd letters from the set $\mathrm{DW}_k^o(s)$;*

having the following properties:

$$\begin{gathered} \operatorname{tot} c = \operatorname{tot} c' + \operatorname{tot} v^e + \operatorname{tot} v^o; \quad \operatorname{neg} w = \operatorname{neg} w' + \lambda(v^o); \\ \operatorname{fix} w^+ = \lambda(v^e); \quad \operatorname{fix}^- w = \lambda(v^o). \end{gathered} \tag{4.1}$$

The bijection $\binom{c}{w} \mapsto (\binom{c'}{w'}, v^e, v^o)$ is quite natural to define. Only its reverse requires some attention. To get the latter three-term sequence from $\binom{c}{w}$ proceed as follows:

(a) let $l_1, \ldots, l_\alpha$ (resp. $m_1, \ldots, m_\beta$) be the increasing sequence of the integers l_i (resp. m_i) such that x_{l_i} (resp. x_{m_i}) is a positive (resp. negative) fixed point of w;

(b) define: $v^e := c_{l_1} \cdots c_{l_\alpha}$ and $v^o := c_{m_1} \cdots c_{m_\beta}$;

(c) remove all the columns $\binom{c_{l_1}}{x_{l_1}}, \ldots, \binom{c_{l_\alpha}}{x_{l_\alpha}}, \binom{c_{m_1}}{x_{m_1}}, \ldots, \binom{c_{m_\beta}}{x_{m_\beta}}$ from $\binom{c}{w}$ and let c' be the nonincreasing word derived from c after the removal;

(d) once the letters $x_{l_1}, \ldots, x_{l_\alpha}, x_{m_1}, \ldots, x_{m_\beta}$ have been removed from the signed permutation w the remaining ones form a signed permutation of a subset A of $[n]$, of cardinality $n - \alpha - \beta$. Using the unique increasing bijection ϕ of A onto the interval $[n - \alpha - \beta]$ replace each remaining letter x_i by $\phi(x_i)$ if $x_i > 0$ or by $-\phi(-x_i)$ if $x_i < 0$. Let w' be the signed derangement of order $n - \alpha - \beta$ thereby obtained.

For instance, with the above weighted signed permutation we have: $v^e = 10, 10, 4, 4$ and $v^o = 7, 3$. After removing the fixed point columns we obtain:

$$\left(\begin{array}{c|cc|c|cc|c} 3 & 4 & 6 & 7 & 11 & 12 & 13 \\ 9 & 7 & 7 & 4 & 2 & 2 & 1 \\ \bar{7} & \bar{6} & \bar{4} & 3 & 12 & 13 & \overline{11} \end{array}\right) \text{ and then } \begin{pmatrix} c' \\ w' \end{pmatrix} = \left(\begin{array}{c|cc|c|cc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 9 & 7 & 7 & 4 & 2 & 2 & 1 \\ \bar{4} & \bar{3} & \bar{2} & 1 & 6 & 7 & \bar{5} \end{array}\right).$$

There is no difficulty verifying that the properties listed in (4.1) hold. For reconstructing $\binom{c}{w}$ from the sequence $(\binom{c'}{w'}, v^e, v^o)$ consider the nonincreasing rearrangement of the juxtaposition product $v^e v^o$ in the form $b_1^{h_1} \cdots b_m^{h_m}$, where $b_1 > \cdots > b_m$ and $h_i \geq 1$ (resp. $h_i = 1$) if b_i is even (resp. odd). The pair $\binom{c'}{w'}$ being decomposed into matrix blocks, as shown in the example, each letter b_i indicates where the h_i fixed point columns are to be inserted. We do not give more details and simply illustrate the construction with the running example.

With the previous example $b_1^{h_1} \cdots b_m^{h_m} = 10^2\, 7\, 4^2\, 3$. First, implement 10^2:

$$\left(\begin{array}{cc|c|cc|c|cc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \mathbf{10} & \mathbf{10} & 9 & 7 & 7 & 4 & 2 & 2 & 1 \\ 1 & 2 & \bar{6} & \bar{5} & \bar{4} & 3 & 8 & 9 & \bar{7} \end{array}\right);$$

then 7:

$$\left(\begin{array}{cc|c|ccc|c|cc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 10 & 9 & 7 & \mathbf{7} & 7 & 4 & 2 & 2 & 1 \\ 1 & 2 & \bar{7} & \bar{6} & \bar{5} & \bar{4} & 3 & 9 & 10 & \bar{8} \end{array}\right);$$

notice that because of condition (*wsp3*) the letter **7** is to be inserted in *second* position in the third block;

then insert 4^2:

$$\left(\begin{array}{cc|c|ccc|ccc|cc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 10 & 10 & 9 & 7 & 7 & 7 & 4 & \mathbf{4} & \mathbf{4} & 2 & 2 & 1 \\ 1 & 2 & \overline{7} & \overline{6} & \overline{5} & \overline{4} & 3 & 8 & 9 & 11 & 12 & \overline{10} \end{array}\right).$$

The implementation of 3 gives back the original weighted signed permutation $\binom{c}{w}$.

5. Proof of Theorem 1.3

It is q-routine (see, e.g., [An76, chap. 3]) to prove the following identities, where v_1 is the first letter of v:

$$\frac{1}{(u;q)_N} = \sum_{n\ge 0} \begin{bmatrix} N+n-1 \\ n \end{bmatrix}_q u^n; \qquad \begin{bmatrix} N+n \\ n \end{bmatrix}_q = \sum_{v\in \mathrm{NIW}_n(N)} q^{\mathrm{tot}\, v};$$

$$\frac{1}{(u;q)_{N+1}} = \sum_{n\ge 0} u^n \sum_{v\in \mathrm{NIW}_n(N)} q^{\mathrm{tot}\, v} = \frac{1}{1-u} \sum_{v\in \mathrm{NIW}_n} q^{\mathrm{tot}\, v} u^{v_1};$$

$$\frac{1}{(u;q^2)_{\lfloor s/2\rfloor +1}} = \sum_{n\ge 0} u^n \sum_{v^e\in \mathrm{NIW}_n^e(s)} q^{\mathrm{tot}\, v^e}; \tag{5.1}$$

$$(-uq;q^2)_{\lfloor (s+1)/2\rfloor} = \sum_{n\ge 0} u^n \sum_{v^o\in \mathrm{DW}_n^o(s)} q^{\mathrm{tot}\, v^o}. \tag{5.2}$$

The last two formulas and Proposition 4.1 are now used to calculate the generating function for the weighted signed permutations. The symbols $\mathrm{NIW}^e(s)$, $\mathrm{DW}^o(s)$, $\mathrm{WSP}(s)$, $\mathrm{WSD}(s)$ designate the unions for $n\ge 0$ of the corresponding symbols with an n-subscript.

Proposition 5.2. *The following identity holds:*

$$\begin{aligned} &\sum_{n\ge 0} u^n \sum_{\binom{c}{w}\in \mathrm{WSP}_n(s)} q^{\mathrm{tot}\, c} Y_0^{\mathrm{fix}^+ w} Y_1^{\mathrm{fix}^- w} Z^{\mathrm{neg}\, w} \\ &\quad = \frac{(u;q^2)_{\lfloor s/2\rfloor+1}}{(uY_0;q^2)_{\lfloor s/2\rfloor+1}} \frac{(-uqY_1Z;q^2)_{\lfloor (s+1)/2\rfloor}}{(-uqZ;q^2)_{\lfloor (s+1)/2\rfloor}} \times \sum_{n\ge 0} u^n \sum_{\binom{c}{w}\in \mathrm{WSP}_n(s)} q^{\mathrm{tot}\, c} Z^{\mathrm{neg}\, w}\ . \end{aligned} \tag{5.3}$$

Proof. First, summing over $(w^e, w^o, \binom{c}{w}) \in \mathrm{NIW}(s)\times \mathrm{DW}(s)\times \mathrm{WSP}(s)$, we have

$$\begin{aligned} &\sum_{w^e,w^o,\binom{c}{w}} u^{\lambda(w^e)} q^{\mathrm{tot}\, w^e} \times (uZ)^{\lambda(w^o)} q^{\mathrm{tot}\, w^o} \times u^{\lambda(c)} q^{\mathrm{tot}\, c} Y_0^{\mathrm{fix}^+ w} Y_1^{\mathrm{fix}^- w} Z^{\mathrm{neg}\, w} \\ &\qquad = \frac{(-uqZ;q^2)_{\lfloor (s+1)/2\rfloor}}{(u;q^2)_{\lfloor s/2\rfloor+1}} \times \sum_{\binom{c}{w}} u^{\lambda(c)} q^{\mathrm{tot}\, c} Y_0^{\mathrm{fix}^+ w} Y_1^{\mathrm{fix}^- w} Z^{\mathrm{neg}\, w} \end{aligned} \tag{5.4}$$

by (5.1) and (5.2). Now, Proposition 4.1 implies that the initial expression can also be summed over five-term sequences $(\binom{c'}{w'}, v^e, v^o, w^e, w^o)$ from $\mathrm{WSD}(s) \times \mathrm{NIW}^e(s) \times \mathrm{DW}^o(s) \times \mathrm{NIW}^e(s) \times \mathrm{DW}^o(s)$ in the form

$$\begin{aligned}
&\sum_{\binom{c'}{w'}, v^e, v^o, w^e, w^o} u^{\lambda(c')} q^{\operatorname{tot} c'} Z^{\operatorname{neg} w'} \times (uY_0)^{\lambda(v^e)} q^{\operatorname{tot} v^e} \times (uY_1 Z)^{\lambda(v^o)} q^{\operatorname{tot} v^o} \\
&\qquad\qquad \times u^{\lambda(w^e)} q^{\operatorname{tot} w^e} \times (uZ)^{\lambda(w^o)} q^{\operatorname{tot} w^o} \\
&\quad = \sum_{v^e, v^o} (uY_0)^{\lambda(v^e)} q^{\operatorname{tot} v^e} \times (uY_1 Z)^{\lambda(v^o)} q^{\operatorname{tot} v^o} \\
&\qquad \times \sum_{\binom{c'}{w'}, w^e, w^o} u^{\lambda(c')} q^{\operatorname{tot} c'} Z^{\operatorname{neg} w'} \times u^{\lambda(w^e)} q^{\operatorname{tot} w^e} \times (uZ)^{\lambda(w^o)} q^{\operatorname{tot} w^o}.
\end{aligned}$$

The first summation can be evaluated by (5.1) and (5.2), while by Proposition 4.1 again the second sum can be expressed as a sum over weighted signed permutations $\binom{c}{w} \in \mathrm{WSP}(s)$. Therefore, the initial sum is also equal to

$$\frac{(-uqY_1 Z; q^2)_{\lfloor (s+1)/2 \rfloor}}{(uY_0; q^2)_{\lfloor s/2 \rfloor + 1}} \times \sum_{\binom{c}{w} \in \mathrm{WSP}(s)} u^{\lambda(c)} q^{\operatorname{tot} c} Z^{\operatorname{neg} w}. \tag{5.5}$$

Identity (5.3) follows by equating (5.4) with (5.5). ∎

Proposition 5.3. *The following identity holds:*

$$\sum_{n \geq 0} u^n \sum_{\binom{c}{w} \in \mathrm{WSP}_n(s)} q^{\operatorname{tot} c} Z^{\operatorname{neg} w} = \Bigl(1 - u \sum_{i=0}^{s} q^i Z^{\chi(i \text{ odd})}\Bigr)^{-1}. \tag{5.6}$$

Proof. For proving the equivalent identity

$$\sum_{\binom{c}{w} \in \mathrm{WSP}_n(s)} q^{\operatorname{tot} c} Z^{\operatorname{neg} w} = \Bigl(\sum_{i=0}^{s} q^i Z^{\chi(i \text{ odd})}\Bigr)^n \quad (n \geq 0) \tag{5.7}$$

it suffices to construct a bijection $\binom{c}{w} \mapsto d$ of $\mathrm{WSP}_n(s)$ onto $\{0, 1, \ldots, s\}^n$ such that $\operatorname{tot} c = \operatorname{tot} d$ and $\operatorname{neg} w = \operatorname{odd} d$. This bijection is one of the main ingredients of the *MacMahon Verfahren* for signed permutations that has been fully described in [FoHa05, § 4]. We simply recall the construction of the bijection by means of an example.

Start with $\begin{pmatrix} c \\ w \end{pmatrix} = \begin{pmatrix} 10\,9\,7\,4\,4\,2\,2\,1\,1 \\ 1\,\overline{4}\,\overline{3}\,2\,5\,6\,8\,\overline{9}\,\overline{7} \end{pmatrix}$. Then, form the two-matrix $\begin{pmatrix} 10\,9\,7\,4\,4\,2\,2\,1\,1 \\ 1\,\,4\,3\,2\,5\,6\,8\,9\,7 \end{pmatrix}$, where the *negative* integers on the bottom row have

been replaced by their opposite values. Next, rearrange its columns in such a way that the bottom row is precisely $1\,2\,\ldots\,n$. The word d is defined to be the top row in the resulting matrix. Here $\binom{d}{\mathrm{Id}}=\binom{10\,4\,7\,9\,4\,2\,1\,2\,1}{1\,2\,3\,4\,5\,6\,7\,8\,9}$. As d is a rearrangement of c, we have $\operatorname{tot} c = \operatorname{tot} d$ and $\operatorname{neg} w = \operatorname{odd} d$. For reconstructing the pair $\binom{c}{w}$ from $d = d_1 d_2 \cdots d_n$ simply make a full use of condition (*wsp3*).

Using the properties of this bijection we have:

$$\sum_{\binom{c}{w}\in \mathrm{WSP}_n(s)} q^{\operatorname{tot} c} Z^{\operatorname{neg} w} = \sum_{d\in\{0,1,\ldots,s\}^n} q^{\operatorname{tot} d} Z^{\operatorname{odd} d} = \sum_{d\in\{0,1,\ldots,s\}^n} \prod_{i=1}^{n} q^{d_i} Z^{\chi(d_i\ \mathrm{odd})}$$
$$= \prod_{i=1}^{n} \sum_{d_i\in\{0,1,\ldots,s\}} q^{d_i} Z^{\chi(d_i\ \mathrm{odd})} = \Bigl(\sum_{i=0}^{s} q^i Z^{\chi(i\ \mathrm{odd})}\Bigr)^n. \qquad \square$$

Let $G_n := G_n(t, q, Y_0, Y_1, Z)$ denote the right-hand side of (1.12) in the statement of Theorem 1.3.

Proposition 5.4. *Let $G_n := G_n(t, q, Y_0, Y_1, Z)$ denote the right-hand side of* (1.12) *in the statement of Theorem* 1.3. *Then*

$$\frac{1+t}{(t^2;q^2)_{n+1}}\, G_n = \sum_{s\ge 0} t^s \sum_{\binom{c}{w}\in \mathrm{WSP}_n(s)} q^{\operatorname{tot} c} Y_0^{\operatorname{fix}^+ w} Y_1^{\operatorname{fix}^- w} Z^{\operatorname{neg} w}. \tag{5.8}$$

Proof. A very similar calculation has been made in the proof of Theorem 4.1 in [FoHa05]. We also make use of the identities on the q-ascending factorials that were recalled in the beginning of this section. First,

$$\frac{1+t}{(t^2;q^2)_{n+1}} = \sum_{r'\ge 0} (t^{2r'} + t^{2r'+1}) \begin{bmatrix} n+r' \\ r' \end{bmatrix}_{q^2}$$
$$= \sum_{r\ge 0} t^r \begin{bmatrix} n+\lfloor r/2\rfloor \\ \lfloor r/2\rfloor \end{bmatrix}_{q^2} = \sum_{r\ge 0} t^r \sum_{b\in \mathrm{NIW}_n(\lfloor r/2\rfloor)} q^{2\operatorname{tot} b}.$$

Then,

$$\frac{1+t}{(t^2;q^2)_{n+1}}\, G_n = \sum_{r\ge 0} t^r \sum_{\substack{b\in \mathrm{NIW}_n,\\ 2b_1\le r}} q^{2\operatorname{tot} b} \sum_{w\in B_n} t^{\operatorname{fdes} w} q^{\operatorname{fmaj} w} Y_0^{\operatorname{fix}^+ w} Y_1^{\operatorname{fix}^- w} Z^{\operatorname{neg} w}$$
$$= \sum_{s\ge 0} t^s \sum_{\substack{b\in \mathrm{NIW}_n,\, w\in B_n\\ 2b_1+\operatorname{fdes} w\le s}} q^{2\operatorname{tot} b+\operatorname{fmaj} w} Y_0^{\operatorname{fix}^+ w} Y_1^{\operatorname{fix}^- w} Z^{\operatorname{neg} w}.$$

As proved in [FoHa05, § 4] to each $\binom{c}{w} = \binom{c_1\cdots c_n}{x_1\cdots x_n} \in \mathrm{WSP}_n(s)$ there corresponds a unique $b = b_1 \cdots b_n \in \mathrm{NIW}_n$ such that $2b_1 + \operatorname{fdes} w = c_1$ and

$2\operatorname{tot} b + \operatorname{fmaj} w = \operatorname{tot} c$. Moreover, the mapping $\binom{c}{w} \mapsto (b, w)$ is a bijection of $\mathrm{WSP}_n(s)$ onto the set of all pairs (b, w) such that $b = b_1 \cdots b_n \in \mathrm{NIW}_n$ and $w \in B_n$ with the property that $2b_1 + \operatorname{fdes} w \leq s$.

The word b is determined as follows: write the signed permutation w as a linear word $w = x_1 x_2 \ldots x_n$ and for each $k = 1, 2, \ldots, n$ let z_k be the number of descents $(x_i > x_{i+1})$ in the right factor $x_k x_{k+1} \cdots x_n$ and let ϵ_k be equal to 0 or 1 depending on whether x_k is positive or negative. Also for each $k = 1, 2, \ldots, n$ define $a_k := (c_k - \epsilon_k)/2$, $b_k := (a_k - z_k)$ and form the word $b = b_1 \cdots b_n$.

For example,

$$\begin{array}{rl} \mathrm{Id} &= 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ \ 9\ \ 10 \\ c &= 9\ 7\ 7\ 4\ 4\ 4\ 2\ 2\ \ 1\ \ \ 1 \\ w &= \overline{4}\ \overline{3}\ \overline{2}\ 1\ 5\ 6\ 8\ 9\ \overline{10}\ \overline{7} \\ z &= 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \ 0\ \ \ 0 \\ \epsilon &= 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ \ 1\ \ \ 1 \\ a &= 4\ 3\ 3\ 2\ 2\ 2\ 1\ 1\ \ 0\ \ \ 0 \\ b &= 3\ 2\ 2\ 1\ 1\ 1\ 0\ 0\ \ 0\ \ \ 0 \end{array}$$

Pursuing the above calculation we get (5.8). ∎

We can complete the proof of Theorem 1.3:

$$\begin{aligned}
&\sum_{n\geq 0}(1+t)G_n(t,q,Y_0,Y_1,Z)\frac{u^n}{(t^2;q^2)_{n+1}} \\
&\quad = \sum_{s\geq 0} t^s \sum_{n\geq 0} u^n \sum_{\binom{c}{w}\in \mathrm{WSP}_n(s)} q^{\operatorname{tot} c} Y_0^{\operatorname{fix}^+ w} Y_1^{\operatorname{fix}^- w} Z^{\operatorname{neg} w} && \text{[by (5.8)]} \\
&\quad = \sum_{s\geq 0} t^s \frac{(u;q^2)_{\lfloor s/2\rfloor+1}}{(uY_0;q^2)_{\lfloor s/2\rfloor+1}} \frac{(-uqY_1Z;q^2)_{\lfloor (s+1)/2\rfloor}}{(-uqZ;q^2)_{\lfloor (s+1)/2\rfloor}} \\
&\qquad\qquad \times \sum_{n\geq 0} u^n \sum_{\binom{c}{w}\in \mathrm{WSP}_n(s)} q^{\operatorname{tot} c} Z^{\operatorname{neg} w} && \text{[by (5.3)]} \\
&\quad = \sum_{s\geq 0} t^s \Bigl(1 - u\sum_{i=0}^{s} q^i Z^{\chi(i \text{ odd})}\Bigr)^{-1} \\
&\qquad\qquad \times \frac{(u;q^2)_{\lfloor s/2\rfloor+1}}{(uY_0;q^2)_{\lfloor s/2\rfloor+1}} \frac{(-uqY_1Z;q^2)_{\lfloor (s+1)/2\rfloor}}{(-uqZ;q^2)_{\lfloor (s+1)/2\rfloor}} && \text{[by (5.6)].}
\end{aligned}$$

Hence, $G_n(t,q,Y_0,Y_1,Z) = B_n(t,q,Y_0,Y_1,Z)$ for all $n \geq 0$. ∎

6. Specializations

For deriving the specializations of the polynomials ${}^{\ell}B_n(q,Y_0,Y_1,Z)$ and $B_n(t,q,Y_0,Y_1,Z)$ with their combinatorial interpretations we refer

to the diagram displayed in Fig. 1. Those two polynomials are now regarded as generating polynomials for B_n by the multivariable statistics $(\ell, \mathrm{pix}^+, \mathrm{pix}^-, \mathrm{neg})$ and $(\mathrm{fdes}, \mathrm{fmaj}, \mathrm{fix}^+, \mathrm{fix}^-, \mathrm{neg})$, their factorial generating functions being given by (1.7) and (1.9), respectively.

First, identity (1.8) is deduced from (1.9) by the traditional token that consists of multiplying (1.9) by $(1-t)$ and making $t=1$. Accordingly, $B_n(q, Y_0, Y_1, Z)$ occurring in (1.8) is the generating polynomial for the group B_n by the statistic $(\mathrm{fmaj}, \mathrm{fix}^+, \mathrm{fix}^-, \mathrm{neg})$.

Now, let

$$B(q, Y_0, Y_1, Z; u) := \sum_{n\ge 0} \frac{u^n}{(q^2;q^2)_n} B_n(q, Y_0, Y_1, Z). \tag{6.1}$$

The *involution* of B_n defined by $w = x_1x_2\cdots x_n \mapsto \overline{w} := \overline{x}_1\overline{x}_2\cdots\overline{x}_n$ has the following properties:

$$\mathrm{fmaj}\, w + \mathrm{fmaj}\, \overline{w} = n^2; \quad \mathrm{neg}\, w + \mathrm{neg}\, \overline{w} = n; \tag{6.2}$$

$$\mathrm{fix}^+\, w = \mathrm{fix}^-\, \overline{w}; \quad \mathrm{fix}^-\, w = \mathrm{fix}^+\, \overline{w}. \tag{6.3}$$

Consequently, the duality between positive and negative fixed points must be reflected in the expression of $B(q, Y_0, Y_1, Z; u)$ itself, as shown next.

Proposition 6.1. *We have:*

$$B(q, Y_0, Y_1, Z; u) = B(q^{-1}, Y_1, Y_0, Z^{-1}; -uq^{-1}Z). \tag{6.4}$$

Proof. The combinatorial proof consists of using the relations written in (6.2), (6.3) and easily derive the identity

$$B_n(q, Y_0, Y_1, Z) = q^{n^2} Z^n\, B_n(q^{-1}, Y_1, Y_0, Z^{-1}). \tag{6.5}$$

With this new expression for the generating polynomial identity (6.1) becomes

$$B(q, Y_0, Y_1, Z; u) = \sum_{n\ge 0} \frac{(-uq^{-1}Z)^n}{(q^{-2};q^{-2})_n} B_n(q^{-1}, Y_1, Y_0, Z^{-1}),$$

which implies (6.4).

The analytical proof consists of showing that the right-hand side of identity (1.8) is invariant under the transformation

$$(q, Y_0, Y_1, Z, u) \mapsto (q^{-1}, Y_1, Y_0, Z^{-1}, -uq^{-1}Z).$$

The factor $1 - u(1+qZ)/(1-q^2)$ is clearly invariant. As for the other two factors it suffices to expand them by means of the q-binomial theorem ([GaRa90], p. 7) and observe that they are simply permuted when the transformation is applied. ∎

The polynomial $D_n^B(t,q,Y_1,Z) := B_n(t,q,0,Y_1,Z)$ (resp. $D_n^B(q,Y_1,Z) := B_n(q,0,Y_1,Z)$) is the generating polynomial for the set D_n^B of the *signed derangements* by the statistic $(\mathrm{fdes},\mathrm{fmaj},\mathrm{fix}^-,\mathrm{neg})$ (resp. $(\mathrm{fmaj},\mathrm{fix}^-,\mathrm{neg})$). Their factorial generating functions are obtained by letting $Y_0=0$ in (1.9) and (1.8), respectively.

Let $Y_0=0$, $Y_1=1$ in (1.8). We then obtain the factorial generating function for the polynomials $D_n^B(q,Z) := \sum q^{\mathrm{fmaj}\,w} Z^{\mathrm{neg}\,w}$ $(w\in D_n^B)$ in the form

$$\sum_{n\ge 0}\frac{u^n}{(q^2;q^2)_n}D_n^B(q,Z) = \Bigl(1-u\frac{1+qZ}{1-q^2}\Bigr)^{-1}\times (u;q^2)_\infty. \tag{6.6}$$

It is worth writing the equivalent forms of that identity:

$$\frac{(q^2;q^2)_n}{(1-q^2)^n}\,(1+qZ)^n = \sum_{k=0}^n \begin{bmatrix} n\\ k\end{bmatrix}_{q^2} D_k^B(q,Z)\quad (n\ge 0); \tag{6.7}$$

$$D_n^B(q,Z)=\sum_{k=0}^n \begin{bmatrix} n\\ k\end{bmatrix}_{q^2}(-1)^k q^{k(k-1)}\,\frac{(q^2;q^2)_{n-k}}{(1-q^2)^{n-k}}(1+qZ)^{n-k}\quad (n\ge 0); \tag{6.8}$$

$$\begin{aligned}&D_0^B(q,Z)=1,\quad \text{and for } n\ge 0\\ &D_{n+1}^B(q,Z)=(1+qZ)\frac{1-q^{2n+2}}{1-q^2}D_n^B(q,Z)+(-1)^{n+1}q^{n(n+1)}.\end{aligned} \tag{6.9}$$

$$\begin{aligned}&D_0^B(q,Z)=1,\quad D_1^B(q,Z)=Zq,\quad \text{and for } n\ge 1\\ &D_{n+1}^B(q,Z)=\Bigl(\frac{1-q^{2n}}{1-q^2}+qZ\frac{1-q^{2n+2}}{1-q^2}\Bigr)D_n^B(q,Z)\\ &\qquad\qquad\qquad +(1+qZ)q^{2n}\frac{1-q^{2n}}{1-q^2}D_{n-1}^B(q,Z).\end{aligned} \tag{6.10}$$

Note that (6.8) is derived from (6.6) by taking the coefficients of u^n on both sides. Next, multiply both sides of (6.6) by the second q^2-exponential $E_{q^2}(-u)$ and look for the coefficients of u^n on both sides. This yields (6.7). Now, write (6.6) in the form

$$E_{q^2}(-u) = \Bigl(1-u\frac{1+qZ}{1-q^2}\Bigr)\sum_{n\ge 0}\frac{u^n}{(q^2;q^2)_n}D_n^B(q,Z) \tag{6.11}$$

and take the coefficients of u^n on both sides. This yields (6.9). Finally, (6.10) is a simple consequence of (6.9).

When $Z=1$, formulas (6.6), (6.8), (6.9) have been proved by Chow [Ch06] with $D_n^B(q)=\sum_w q^{\mathrm{fmaj}\,w}$ $(w\in D_n^B)$.

Now the polynomial $K_n^B(q,Y_1,Z) := {}^{\ell}B_n(q,0,Y_1,Z)$ is the generating polynomial for the set K_n^B of the *signed desarrangements* by the statistic $(\ell, \mathrm{pix}^-, \mathrm{neg})$. From (1.7) we get

$$\text{(6.12)}\quad \sum_{n\ge 0} \frac{u^n}{(-Zq;q)_n\,(q;q)_n} K_n^B(q,Y_1,Z) = \Bigl(1-\frac{u}{1-q}\Bigr)^{-1} \times (u;q)_\infty \Bigl(\sum_{n\ge 0} \frac{q^{\binom{n+1}{2}}(Y_1 Z u)^n}{(-Zq;q)_n\,(q;q)_n}\Bigr).$$

When the variable Z is given the zero value, the polynomials in the second column of Fig. 1 are mapped on generating polynomials for the *symmetric group*, listed in the third column. Also the variable Y_1 vanishes. Let $A_n(t,q,Y_0) := B_n(t^{1/2},q^{1/2},Y_0,0,0)$. Then

$$\text{(6.13)}\qquad A_n(t,q,Y_0) = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}\,\sigma} q^{\mathrm{maj}\,\sigma} Y_0^{\mathrm{fix}\,\sigma} \quad (\mathrm{fix} := \mathrm{fix}^+).$$

Identity (1.9) specializes into

$$\text{(6.14)}\quad \sum_{n\ge 0} A_n(t,q,Y_0)\frac{u^n}{(t;q)_{n+1}} = \sum_{s\ge 0} t^s \Bigl(1-u\sum_{i=0}^{s} q^i\Bigr)^{-1} \frac{(u;q)_{s+1}}{(uY_0;q)_{s+1}},$$

an identity derived by Gessel and Reutenauer ([GeRe93], Theorem 8.4) by means of a quasi-symmetric function technique. Note that they wrote their formula for "$1+\mathrm{des}$" and not for "des."

Multiply (6.14) by $(1-t)$ and let $t:=1$, or let $Z:=0$ and q^2 be replaced by q in (1.8). Also, let $A_n(q,Y_0) := \sum_\sigma q^{\mathrm{maj}\,\sigma} Y^{\mathrm{fix}\,\sigma}$ $(\sigma\in\mathfrak{S}_n)$; we get

$$\text{(6.15)}\qquad \sum_{n\ge 0} \frac{u^n}{(q;q)_n} A_n(q,Y_0) = \Bigl(1-\frac{u}{1-q}\Bigr)^{-1} \frac{(u;q)_\infty}{(uY_0;q)_\infty},$$

an identity derived by Gessel and Reutenauer [GeRe93] and also by Clarke *et al.* [ClHaZe97] by means of a q-Seidel matrix approach.

We do not write the specialization of (6.14) when $Y_0 := 0$ to obtain the generating function for the polynomials $D_n(t,q) := \sum\limits_{\sigma\in D_n} t^{\mathrm{des}\,\sigma} q^{\mathrm{maj}\,\sigma}$. As for the polynomial $D_n(q) := \sum\limits_{\sigma\in D_n} q^{\mathrm{maj}\,\sigma}$, it has several analytical expressions, which can all be derived from (6.7)–(6.10) by letting $Z:=0$ and q^2 being replaced by q. We only write the identity which corresponds to (6.7)

$$\text{(6.16)}\qquad D_0(q)=1 \quad\text{and}\quad \frac{(q;q)_n}{(1-q)^n} = \sum_{k=0}^{n} \begin{bmatrix} n\\ k\end{bmatrix}_q D_k(q) \quad \text{for } n\ge 1,$$

which is then equivalent to the identity

$$\text{(6.17)}\qquad e_q(u) \sum_{n\ge 0} \frac{u^n}{(q;q)_n} D_n(q) = \Bigl(1-\frac{u}{1-q}\Bigr)^{-1}.$$

The specialization of (6.8) for $Z := 0$ and q^2 replaced by q was originally proved by Wachs [Wa98] and again recently by Chen and Xu [ChXu06]. Those two authors make use of the now classical *MacMahon Verfahren*, that has been exploited in several papers and further extended to the case of signed permutation, as described in our previous paper [FoHa05].

In the next proposition we show that $D_n(q)$ can be expressed as a polynomial in q with *positive integral* coefficients. In the same manner, the usual derangement number d_n is an explicit sum of *positive* integers. To the best of the authors' knowledge those formulas have not appeared elsewhere.

Proposition 6.2. *The following expressions hold:*

$$D_n(q) = \sum_{2\le 2k\le n-1} \frac{1-q^{2k}}{1-q}\,\frac{(q^{2k+2};q)_{n-2k-1}}{(1-q)^{n-2k-1}}\,q^{\binom{2k}{2}} + q^{\binom{n}{2}}\chi(n \text{ even}), \tag{6.18}$$

$$d_n = \sum_{2\le 2k\le n-1} (2k)(2k+2)_{n-2k-1} + \chi(n \text{ even}). \tag{6.19}$$

Proof. When $q=1$, then (6.18) is transformed into (6.19). As for (6.18), an easy q-calculation shows that its right-hand side satisfies (6.9) when $Z=0$ and q replaced by $q^{1/2}$. ∎

Now, let ${}^{\mathrm{inv}}A_n(q,Y_0) := {}^{\ell}B_n(q,Y_0,0,0)$. Then

$${}^{\mathrm{inv}}A_n(q,Y_0) = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}\,\sigma}\,Y_0^{\mathrm{pix}\,\sigma} \quad (\mathrm{pix} := \mathrm{pix}^+). \tag{6.20}$$

Formula (1.7) specializes into

$$\sum_{n\ge 0} \frac{u^n}{(q;q)_n}\,{}^{\mathrm{inv}}A_n(q,Y_0) = \Bigl(1-\frac{u}{1-q}\Bigr)^{-1} \frac{(u;q)_\infty}{(uY_0;q)_\infty}; \tag{6.21}$$

In view of (6.15) we conclude that

$$A_n(q,Y_0) = {}^{\mathrm{inv}}A_n(q,Y_0). \tag{6.22}$$

For each permutation $\sigma = \sigma(1)\cdots\sigma(n)$ let the *ligne of route* of σ be defined by $\mathrm{Ligne}\,\sigma := \{i : \sigma(i) > \sigma(i+1)\}$ and the *inverse ligne of route* by $\mathrm{Iligne}\,\sigma := \mathrm{Ligne}\,\sigma^{-1}$. Notice that $\mathrm{maj}\,\sigma = \sum_i i\,\chi(i \in \mathrm{Ligne}\,\sigma)$; we also let $\mathrm{imaj}\,\sigma := \sum_i i\,\chi(i \in \mathrm{Iligne}\,\sigma)$. Furthermore, let $\mathbf{i} : \sigma \mapsto \sigma^{-1}$. If Φ designates the *second fundamental transformation* described in [Fo68], [FoSc78], it is known that the bijection $\Psi := \mathbf{i}\,\Phi\,\mathbf{i}$ of $\mathfrak{S}_n$ onto itself has the following property: $(\mathrm{Ligne}, \mathrm{imaj})\,\sigma = (\mathrm{Ligne}, \mathrm{inv})\,\Psi(\sigma)$. Hence,

$$(\mathrm{pix}, \mathrm{imaj})\,\sigma = (\mathrm{pix}, \mathrm{inv})\,\Psi(\sigma) \tag{6.23}$$

and then $A_n(q,Y_0)$ has the other interpretation:

$$A_n(q,Y_0) = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{imaj}\,\sigma} Y_0^{\mathrm{pix}\,\sigma}. \tag{6.24}$$

Finally, let $K_n(q) := \sum_{\sigma\in K_n} q^{\mathrm{inv}\,\sigma}$. Then, with $Y_0 := 0$ in (6.22) we have:

$$K_n(q) = {}^{\mathrm{inv}}A_n(q,0) = A_n(q,0) = D_n(q). \tag{6.25}$$

However, it can be shown directly that $K_n(q)$ is equal to the right-hand side of (6.18), because the sum occurring in (6.18) reflects the geometry of the desarrangements. The running term is nothing but the generating polynomial for the desarrangements of order n whose leftmost trough is at position $2k$ by the number of inversions "inv."

The bijection Ψ also sends K_n onto itself, so that

$$\sum_{\sigma\in K_n} q^{\mathrm{inv}\,\sigma} = \sum_{\sigma\in K_n} q^{\mathrm{imaj}\,\sigma}, \tag{6.26}$$

a result obtained in this way by Désarménien and Wachs [DeWa90, 93], who also proved that for every subset $E \subset [n-1]$ we have

$$\#\{\sigma\in D_n : \mathrm{Ligne}\,\sigma = E\} = \#\{\sigma\in K_n : \mathrm{Iligne}\,\sigma = E\}. \tag{6.27}$$

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